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GRAPHICAL AND THEORETICAL ANALYSIS OF STEP-DRAWDOWN TEST OF ARTESIAN WELL[1]

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ABSTRACT

The drawdown in an artesian well resulting from the withdrawal of water is made up of head loss resulting from laminar flow in the formation, and head loss resulting from turbulent flow in the zone outside the well, through the well screen, and in the well casing.

A graphical method for the empirical determination of laminar and turbulent head losses is given in this paper. The empirical method is compared and related to theoretical equations. A method is given for computing the head-loss distribution outside the pumped well for various pumping rates. Analysis is made of the variation of specific capacity with discharge and of the importance of well radius in well design.

INTRODUCTION

C. E. Jacob, in a paper entitled, "Drawdown test to determine effective radius of artesian well,"[3] presented a method for evaluating the head loss resulting from turbulent flow in the immediate vicinity of an artesian well by means of a step-drawdown test. The assumption was made that the flow near the well face would be turbulent, the head loss in the zone of turbulence being "approximately proportional to some higher power of the velocity approaching the square of the velocity"; whereas the flow in the remainder of the aquifer would be laminar, the aquifer loss varying directly as the discharge.

N. J. Lusczynski, in a study of which the results have not been published, developed a method using an observation well to eliminate correction for time effects. In that paper, the turbulent-flow exponent was considered as an unknown constant rather than approximated as a square relationship.

Graphical Method

Jacob used the equation

$$s_w = BQ + CQ^2 \tag{1}$$

in which s_w is the drawdown in the pumped well; B, the aquifer constant; C, the "well-loss" constant; and Q, the discharge.[4] As used in this paper, s_w is the drawdown in the pumped well for a constant discharge at a fixed

1. Published by permission of the Director, U. S. Geological Survey.
2. Dist. Engr., U. S. Geol. Survey, Ground Water Branch, Louisville, Ky.
3. Jacob, C. E., Drawdown test to determine effective radius of artesian well: Am. Soc. Civil Eng. Trans., vol. 112, paper no. 2321, p. 1047, 1947.
4. The letter symbols in this paper are defined where they first appear and are assembled alphabetically, for convenience of reference, in the Appendix.

time, pro d the well was in a static condition prior to each step. If the well is not allowed to recover between steps (changes in pumping rates), drawdown curves must be extended as was done by Jacob. In this paper s_w for any step is equal to the sum of the incremental drawdowns used by Jacob, $s_w = \Sigma(\Delta s_1 + \Delta s_2 + \text{-----} n)$. Jacob's equation can be put in the form

$$\frac{s_w}{Q} - B = CQ$$

or

$$\log\left(\frac{s_w}{Q} - B\right) = \log C + \log Q \qquad (2)$$

The latter is in the form of a straight-line equation and will plot as a straight line on log-log paper when Q is plotted against $(\frac{s_w}{Q} - B)$. At Q = 1, $C = \frac{s_w}{Q} - B$. The slope of the line is unity. On figure 1, line A is plotted from Jacob's solution (B = 19 sec./ft.², C = 1.35 sec.²/ft.⁵) by drawing a straight line at a 45° angle (slope = 1) through the point Q = 1, $\frac{s_w}{Q} - B = C = 1.35$. When the step-test data used by Jacob are plotted on this diagram, it is found that the data for the various steps plot a curve (parameter B = 19 on fig. 1) rather than a straight line. If the exponent for turbulent flow is allowed to stand as the square, points 2, 3, and 4 could be considered to be within the range of error of measurement, although these points indicate possible curvature. Point 1, determined at the lowest rate of discharge, might be discounted if it be assumed that turbulence was not fully developed at that rate.

If the exponent for turbulent flow is expressed as an unknown constant "n," a similar analysis may be used. The equation is then

$$s_w = BQ + CQ^n \qquad (3)$$

which rearranges to

$$\left(\frac{s_w}{Q} - B\right) = CQ^{n-1}$$

or

$$\log\left(\frac{s_w}{Q} - B\right) = \log C + (n-1)\log Q \qquad (4)$$

An empirical solution is obtained by a graphical solution of the straight-line equation on log-log paper.

Assume values of B; for each value of B, plot $(\frac{s_w}{Q} - B)$ vs. Q on log paper. The solution is obtained when a value of B is found that produces a straight line. C is obtained from the intercept; i.e., where Q = 1, $C = \frac{s_w}{Q} - B$. The slope of the line equals (n - 1), which produces a solution for n. Figure 1, based on Jacob's data, illustrates this method. Curves are shown for various trials of B. The final solution is B = 20.4 sec./ft.²; C, from the Q intercept, = 0.44; and n, from the slope, equals 2.64. Units of CQ^n must be feet, so that the units of C are sec.^{n}/ft.$^{3n-1}$.

Table 1 shows original data and drawdown computed by both methods. Also shown are drawdowns for higher rates of pumping as might be computed for design of a pump.

TABLE 1

Comparison of observed drawdown and drawdown computed by two methods

Step	Discharge Q (cfs)	Observed drawdown s_w (feet)	$\frac{s_w}{Q}$ (sec./ft.²)	For B = 20.4 $(\frac{s_w}{Q} - B)$ (sec./ft.²)	Jacob BQ (ft.)	Jacob CQ^2 (ft.)	Jacob s_w (ft.)	Log-log BQ (ft.)	Log-log CQ^n (ft.)	Log-log s_w (ft.)
1	1.21	25.4	21.0	0.6	23.0	2.0	25.0	24.7	0.7	25.4
2	2.28	50.4	22.1	1.7	43.3	7.0	50.3	46.5	3.9	50.4
3	3.12	72.3	23.2	2.8	59.2	13.2	72.4	63.6	8.9	72.5
4	3.39	80.3	23.7	3.3	64.4	15.5	79.9	69.1	11.0	80.1
	4				76.0	21.6	97.6	81.6	17.1	98.7
	5				95.0	33.8	128.8	102.0	30.8	132.8
	6				114.0	48.6	162.6	122.4	50.2	172.6

Inspecti‑ of figure 1 and table 1 shows the following:

1. Draw‑‑wns computed by either method are very close to the original data. The problem is one of writing the equation of a curve fitting four points. The data cover such a short range that any number of different curves could be applied. If the test covers the range of use, either solution may be applied. However, if computations must be made for discharge greatly in excess of those used in the test (for pump design or for determination of maximum yield), interpretation might be poor. Until better methods are devised, a margin of safety can be provided by computation by both methods, using the least favorable solution for the purpose.

2. Comparison of the formational portion of the drawdown (BQ) computed by the two methods shows that the log-log graphical method assigns a larger portion of the total drawdown to formational loss, and shows smaller "well loss" in the range of the data but greater "well loss" in the extension to higher discharges.

3. The graphical method emphasizes the need for accurate data. Small erros in discharge or drawdown measurements will cause relatively large errors in n, B, and C. This sensitivity is present in Jacob's and Lusczynski's methods but is not nearly so apparent as in the log-log graphical method.

4. The graphical method has a definite advantage over the mathematical equation. In applying the equation no allowance can be made for experimental error because the data must be treated in groups. Averaging results obtained by computing successive groups of data may yield poor results. By the graphical method a line is fitted to all the data. If one set of data is poor, it will fall off the curves and can be given less weight. In the mathematical method one poor reading could affect three successive sets of computations. It is generally recognized that graphical procedures are best for treating problems based on measurements of physical quantities in which errors are inherent. Rounding or adjusting the original data graphically before applying the equation may eliminate poor data; however, such a procedure must be done cautiously as adjusted data may lead to misinterpretation.

5. The graphical method is much less complicated than the mathematical. It can be applied treating the exponent as an unknown constant; or, if conditions warrant acceptance of the exponent as a square, the best fit of a 45° line will give a solution.

Magnitude of exponent "n"

Jacob's adoption of an exponent of 2 for turbulent flow is reasonable in view of the results of widely different hydraulic studies, and it may eventually prove to be an assumption that will produce usable results for many problems.

R. W. Stallman, Lusczynski, Jacob, W. F. Guyton, and the author have discussed this problem for several years. The graphical method has been applied to several tests and in each case the exponent has been greater than 2. Apparently, the explanation lies in the fact that past analysis has been based on the assumption that the critical radius is constant as discharge varies. It seems logical to assume that, at low rates of flow, no turbulence is present in the aquifer. As discharge is increased turbulent flow occurs first at the well face, and as discharge is further increased the boundary between laminar flow and turbulent flow moves outward from the well. The assumption that the head loss can be separated into BQ, representing laminar-flow losses, and CQ^n, representing turbulent-flow losses, for all discharge rates is therefore erroneous; however, the empirical solution is usable for certain problems, as will be demonstrated later in this paper.

Figure 2 is a sketch showing drawdowns near a pumped well for several rates of discharge if the area of turbulence be allowed to vary with d‑ ‑harge. Shown also is the $BQ + CQ^n$ empirical distribution of head. The gra‑ ‑al method is based on the assumption of a constant radius, r_B, for the boundary between "well loss" and formation loss. Only at one discharge rate, Q_B, would the radius r_B coincide with the true boundary between laminar and turbulent flow. At a higher discharge rate the drawdown attributed to formational loss, BQ, carries the logarithmic distribution of head into the turbulent zone as shown by the dashed line in the sketch. The term for "well loss," CQ^n, represents the loss from the intersection of the logarithmic distribution curve with the radius r_B to the pumping level in the well. It includes the loss caused by turbulent flow between r_B and the well face (which varies approximately as the square of the discharge), plus the excess of actual turbulent-flow loss over assumed laminar-flow loss in the zone between r_B and the turbulent-laminar-flow boundary, plus the loss due to upward flow in the well, s_f (which also varies approximately as the square of the discharge). It can be seen, then, that the exponent in the term CQ^n is not a true expression of the variation of head loss with discharge for conditions of turbulent flow. The exponent may be unity at very low rates of discharge, or it may be in excess of 2 for the case where the turbulent zone has a radius in excess of r_B. Jacob's assumption of the exponent of 2 would be true only for the discharge rate of Q_B; however, his assumption would yield a usable solution in cases where the turbulent flow up the well was large relative to the turbulent loss in the aquifer.

If laminar flow exists at low rates, the data should plot horizontally (slope of $n - 1 = 0$) on the log plot. Figure 3 shows data for a step test run southwest of Louisville, Ky., in 1945. The first two points plot as a horizontal line (slope = 0) indicating laminar flow. Since only two points are available for higher rates of discharge, it is not possible to evaluate the test. If the square relationship be assumed, the data may be interpreted as:

$$\text{For } Q < Q_c,\ s_w = BQ + C'Q \text{ (for laminar flow)}$$

$$\text{For } Q > Q_c,\ s_w = BQ + CQ^2 \text{ (for turbulent flow)}$$

where C is the "well-loss" constant for turbulent flow and C' the "well-loss" constant for laminar flow, and Q_c is the critical discharge below which laminar flow prevails. The critical discharge, Q_c, is determined approximately as 1.5 cfs from the intersection of the two straight lines. B is determined as 14.25; C = 1.54 from the intercept of the 45° line; and C' = 2.30 from the intercept of the horizontal line.

Reynolds numbers are computed[5] at the well face for the four steps as 9, 13, 17, and 23. These figures are necessarily approximate, but they do indicate that the assumption of laminar flow for the lower steps is not unreasonable. Tolman[6] indicates that laminar flow exists for $R < 10$.

5. Grain size determined initially by sieve analysis and modified by development of the well during construction to remove the finer-grained 60 percent of the particles. Reynolds number then computed as average grain size of remaining 40 percent of material times velocity times density divided by absolute viscosity. Velocity is used as discharge divided by cross-sectional area at screen.
6. Tolman, C. F., Ground water, p. 199, New York and London, McGraw Hill Book Co., Inc., 1937.

Figure 4 shows a solution of a test run northeast of Louisville in 1946. It is noted that the exponent is greater than 2, and that, if the square is used, the data plot a curve relative to the 45° straight-line solution. Graphical solution of a step test made by Lusczynski at Port Washington, N. Y., in 1947,[7] yields the solution, $s_w = 51.5\,Q + 10.5\,Q^{2.56}$. It may be significant that values of n are consistently in the range near 2.5. (Test by Jacob at Bethpage, Long Island, N. Y., in 1943, 2.64; Louisville test, 2.54; Port Washington test, 2.56; and tests at Houston, Tex., in 1949, 2.43 and 2.82.)

Development of Equation for Separating Laminar and Turbulent Head Losses

The following discussion is given in an attempt to clarify the use of the empirical equation ($s_w = BQ + CQ^n$), to shed some light on the limits of its use and to tie the empirical equation to theory.

The use of "effective radius" is a convenient but somewhat misleading method of handling well-loss problems. Many unknowns or indeterminate factors are thrown into this term. Effects of partial penetration, effects of changes in transmissibility near the well because of development, and many other factors only partly accounted for are included in the term. Since the physical well radius is usually known, this discussion will be based on the inside radius of the well or screen, r_n.

Referring again to figure 2, the drawdown in the well is expressed as

$$s_w = M - N + s_t + s_f \tag{5}$$

in which M represents the head loss at the well face according to laminar-flow theory, N is the head loss according to laminar-flow theory between the critical boundary and the well face, s_t is the head loss according to turbulent-flow theory between the critical boundary and the well face, and s_f is the head loss due to pipe friction in the well.

For artesian conditions, for a fully penetrating well, and after pumping at a constant rate for a time long enough to establish a steady-state condition, the difference in drawdown between two points in the aquifer r_1 and r_2 distant from the pumped well may be expressed

$$s_1 - s_2 = \frac{2.3\,Q \log \frac{r_2}{r_1}}{2\pi T} \tag{6}$$

where Q is the pumping rate, T the transmissibility, and s_1 and s_2 the drawdowns at points r_1 and r_2.

The term M at a fixed time, under equilibrium conditions, (assume homogeneous material and neglect effects of development) is expressed as

$$M = \frac{2.3\,Q \log \frac{r_e}{r_n}}{2\pi T} \tag{7}$$

where r_e is the intercept for zero drawdown on the semilog profile plot and r_n the inside radius of the screen.

7. Data in open file of the U. S. Geological Survey at Washington, D. C., and Mineola, N. Y.

The term N is expressed

$$N = \frac{2.3\,Q \log \frac{r_t}{r_n}}{2\pi T} \tag{8}$$

where r_t is the distance from the well center to the boundary between laminar and turbulent flow.

The distance to the boundary of the zone of turbulent flow (under the conditions set forth) should vary directly with discharge.

$$\frac{r_{t_1}}{Q_1} = \frac{r_{t_2}}{Q_2} \tag{9}$$

At low rates of flow no turbulent flow will exist in the aquifer or screen. The critical discharge at which turbulence is introduced at the well face is designated Q_c. The expression then is written

$$\frac{r_n}{r_t} = \frac{Q_c}{Q} \tag{10}$$

which applies only when $Q > Q_c$.

The expression for N may be modified by substitution to

$$N = \frac{2.3\,Q \log \frac{Q}{Q_c}}{2\pi T} \tag{11}$$

The term s_t, representing the head loss in the turbulent-flow zone between the laminar-turbulent boundary (r_t) and the inside face of the well (r_n), is approximated as follows:

Assume that the loss in head varies as the square of the velocity. The head loss across an increment dr is then expressed $ds = Ev^2 dr$, where E is the aquifer constant for turbulent flow. The velocity at any point is equal to discharge divided by area $v = Q/a$, and area equals $2\pi r m\theta$, in which m represents the height of the section through which flow is occurring, and θ is the porosity of the bed.

Substituting

$$ds = \frac{EQ^2 dr}{(2\pi r m\theta)^2}$$

Integrating between the limits r_t and r_n, we obtain

$$s_t = \frac{EQ^2}{(2\pi m\theta)^2}\left(\frac{1}{r_n} - \frac{1}{r_t}\right) \tag{12}$$

Substituting $r_t = \dfrac{Q\,r_n}{Q_c}$

$$s_t = \frac{EQ^2}{(2\pi m\theta)^2}\left(\frac{1}{r_n} - \frac{Q_c}{Q\,r_n}\right)$$

Simplifying

$$s_t = \frac{EQ}{(2\pi m\theta)^2\,r_n}(Q - Q_c) \tag{13}$$

For a given well the term $E/(2\pi m\theta)^2\,r_n$ may be replaced by a constant D

$$s_t = DQ\,(Q - Q_c) \tag{14}$$

The ter s_f, representing the friction loss caused by flow up the pipe, may be a ied to vary approximately as the square of the discharge

$$s_f = FQ^2 \tag{15}$$

in which F is a constant for friction loss due to turbulent flow up the well.

Bringing the various terms of head loss (equations 7, 8, 14, and 15) together, the equation for the drawdown in the pumped well is

$$s_w = \frac{2.3\,Q}{2\,\pi\,T} \log \frac{r_e}{r_n} - \frac{2.3\,Q}{2\,\pi\,T} \log \frac{Q}{Q_c} + DQ\,(Q - Q_c) + FQ^2 \tag{16}$$

At least one observation well is required in order to apply this equation. The term r_e can be computed if S and T are determined from a time-drawdown plot at one observation well, or can be obtained from the zero-drawdown intercept of a semilog profile plot if several observation wells are available. Q and s_w are measured during the step test and r_n is known.

Unknowns are D, F, and Q_c. If the step test is run at both high and low rates of discharge and if sufficient points are available, Q_c may be determined from the log-log plot as discussed in the first part of this paper and illustrated by figure 3. Q_c is obtained from the intersection of the horizontal straight line for laminar flow and the angular straight line for turbulent flow. If data cannot be obtained at low rates, Q_c cannot be obtained by that method.

Inspection of equation (16) shows that it is impossible to solve for D and F. However, the equation is useful for certain specific cases.

If the equation is written for each of two pumping rates Q_1 and Q_2 and the first equation subtracted from the second, the following equation is obtained:

$$\frac{s_2}{Q_2} - \frac{s_1}{Q_1} = \frac{2.3}{2\,\pi\,T} \log \frac{Q_1}{Q_2} + D\,(Q_2 - Q_1) + F\,(Q_2 - Q_1) \tag{17}$$

For the case of a very deep well where the pipe friction is large compared to the turbulent losses in the aquifer, the term D might be assumed as zero, and in this case an approximate solution for F would be possible. For the case of a well where the pump intake is at or near the screen the pipe losses are very small; in this case the term F may be considered zero, which makes a solution for D possible. For the case where both pipe loss and turbulent loss in the aquifer are substantial, the pipe loss, FQ^2 in equation (16), might be computed from tables of head loss in pipes. The term FQ^2 could also be obtained from field measurements by installing a measuring pipe extending down to the screen.

Application of Theoretical Method

An illustration of the application of equation (16) to test data is given below. The data used are from the test made northeast of Louisville in 1946. The test was made on a 12-inch well screened in the lower 30 feet of an artesian aquifer averaging 67 feet thick and affected by induced infiltration from the Ohio River. For this case the boundary of zero drawdown is a line source located 400 feet from the well. The effective distance to the external boundary as it relates to drawdown at the well is twice the distance to the line source, or r_e = 800 feet; the inside radius of the screen, r_n = 0.45 foot; the transmissibility determined from pumping-test data at observation wells = 120.000 gpd/ft. = 0.185 ft.²/sec.

So far as is known, no solution is available for determining partial-penetration corrections for problems involving both laminar and turbulent flow.

In order to illustrate the use of the theory presented above, the test ta have been converted to the case of full penetration by the application of   ny's[8] approximate equation.

TABLE 2

Test data adjusted for partial penetration

Step	Q (cfs)	s_w a/ (feet)
1	0.819	5.28
2	1.150	7.74
3	1.587	11.33
4	1.961	14.81

a/ One-hour drawdowns equal the sum of the incremental drawdowns caused by each change in pumping rate.

Equation (17) was solved for D by the substitution of test data. Inasmuch as the pump intake was at the top of the screen, the pipe-loss term $F(Q_2 - Q_1)$ was considered to be very small and was neglected. Values of D were determined from steps 1 and 2 as 1.73; steps 2 and 3 as 1.57; and steps 3 and 4 as 1.59. The average value of 1.63 sec.²/ft.⁵ was adopted.

Equation (16) was solved for Q_c for each step (again neglecting the pipe-flow term, FQ^2). Values for Q_c are 0.77, 0.70, 0.76, and 0.78; the average value is 0.75 cfs.

Comparison of Theoretical and Empirical Methods

Solution of the same test data (corrected to case of full penetration) by the log-log graphical method according to the empirical equation

$$s = BQ + CQ^n$$

yields the following: B = 6.05 sec./ft.²; n = 2.54; C = 0.538 sec.$^{2.54}$/ft.$^{6.62}$.

Referring to the sketch, figure 2, we may write

$$BQ = \frac{2.3\,Q}{2\,\pi\,T} \log \frac{r_e}{r_B} \tag{18}$$

Solving this equation yields the value of r_B = 0.69 foot.

From equation (9) we may write

$$\frac{r_B}{Q_B} = \frac{r_n}{Q_c} \tag{19}$$

from which Q_B = 1.15 cfs.

The CQ^n term (see figure 2) may also be set equal to its theoretical equivalent:

$$CQ^n = DQ\,(Q - Q_c) - \frac{2.3\,Q}{2\,\pi\,T} \log \frac{Q}{Q_B} + FQ^2 \tag{20}$$

8. Kozeny (Wasserkraft und Wasserwirtschaft, vol. 28, p. 101); equation is quoted by Morris Muskat in The flow of homogeneous fluids through porous media, p. 274, New York, N. Y., McGraw-Hill Book Company, Inc., 1937.

Neglecting t' FQ^2 term as before, we have an expression equating the empirical exp ion for "well loss" to its theoretical equivalent. All terms in the equation have been determined. It is found that the equation is not mathematically sound. The trouble may be in the assumption that all the losses between the radius r_B and the well can be expressed empirically as CQ^n; should C or n perhaps be treated as variables? Or, perhaps, the trouble may be in the term D in the equation. This term, the constant for turbulent flow in the aquifer for a given well, neglects the effect of development of the well. Variation in D might be expected as the turbulent zone is expanded through the developed zone and into the undisturbed aquifer.

It must also be recognized that equation (14) was developed on the basis of the assumption that head loss in the turbulent zone varies as the square of the velocity. Since the Reynolds curve shows a transition from laminar to turbulent flow, equation (14) is an over simplification of a complex relationship. An attempt was made to include a more accurate representation for equation (14). Reynolds number times friction factor versus Reynolds number were plotted on log-log paper. A constant was subtracted by trial until a straight line was obtained. The equation of this solution was integrated between r_t and r_n. The resulting expression when introduced in equation (16) complicated the equation to the extent that it became unusable for the problem under consideration.

In order to study the degree of error in the equation, values of D were computed for various discharge rates, using the computed values of all other terms in the equation. Figure 5 shows a plot of the values of D. For values of Q up to 5 cfs, values of D required to make the theoretical and empirical (log-log) methods agree are within a few percent of the average value of 1.63 determined by theory, so that extension of the graphical solution appears to correspond very nearly to theory in this range of discharge for this case. At low rates of Q, that is, at rates just above the critical discharge Q_c, the equation is very sensitive, and values of D relating the theoretical and empirical methods are unreliable. However, in this range the error in total drawdown will be small inasmuch as the magnitude of the "well loss" is small.

Figure 6 shows the variation of drawdown with discharge for the example. Curve A represents the formational head loss outside the turbulent zone, computed from equation (16). Curve B is the total drawdown in the well obtained by adding the theoretical loss in the turbulent zone to curve A. Curve C is the head loss assigned to the formation by the graphical method (BQ), and curve D is the total drawdown in the well by the graphical method $s_W = BQ + CQ^n$. It should be noted that curves B and D agree very closely. This comparison indicates that the empirical graphical method gives a total drawdown consistent with theory, for the case demonstrated, so that the graphical method may be applied to problems of pump design and determination of maximum yield of a given well. The graphical method is not applicable for solution of head distribution outside the well, or for determination of design well radius, as shown by the disagreement of lines A and C.

Distribution of Head Loss in the Aquifer

The distance to the boundary between laminar and turbulent flow may be approximated for various discharge rates from the relation $\frac{r_n}{Q_c} = \frac{r_t}{Q}$ (from equation 10).

In figure 7 are shown profiles of head loss for various rates of discharge, and the extent of the turbulent zone for each rate, for the example used.

The curve of head loss through the turbulent-flow zone is compu' 'rom the equation

$$s_t = DQ^2 r_n \left(\frac{1}{r} - \frac{1}{r_t}\right) \qquad (21)$$

Inspection of figure 7 shows the development of the turbulent-flow zone as discharge increases. For discharges less than critical (Q_c = 0.75 cfs in this case) the distribution of head loss in the formation follows the normal variation ($\log \frac{1}{r}$). At higher discharges the logarithmic distribution applies outside the boundary between laminar and turbulent flow. In the turbulent zone the head loss changes at a much greater rate. Note that for a discharge of 5 cfs the head loss from the external boundary to the critical zone (r_t = 3.00 feet) is 23.9 feet, whereas the loss through the turbulent zone (from r_t = 3.00 feet to r_n = 0.450 foot) is 34.7 feet. An 18-inch well (r_n = 0.656) would reduce the turbulent loss from 34.7 feet to 21.9 feet, and a 24-inch well (r_n = 0.906) would have a corresponding turbulent loss of only 14.0 feet.

This diagram, which must be recognized as showing only approximate relationships, indicates that the well radius may be more important in well design than has been considered in the past. Textbooks now in circulation state that the well radius is not very important because the discharge varies as $\log \frac{r_e}{r_n}$, a term that varies little as r_n is changed but that is based on the assumption of laminar flow all the way to the well. This paper demonstrates that the well radius becomes more and more important as discharge increases.

The variation of specific capacity with Q is shown in figure 8 for wells of various diameters at the site of the Louisville test. This figure shows a rapid decline in specific capacity as the discharge is increased beyond the critical discharge. The figure also shows the effect of well radius. For example, at 0.5 cfs the specific capacities of a 12-inch and an 18-inch well are 0.156 and 0.164 cfs/ft., respectively, or a difference of less than 5 percent; at 4 cfs the specific capacities are 0.097 and 0.122 cfs/ft., or a difference of more than 20 percent.

Well Efficiency

In figure 9 are shown "well-efficiency" curves for 12-inch, 18-inch, and 24-inch wells pumped at various rates. For this plotting "well efficiency" is defined as the ratio of (1) the theoretical drawdown computed by assuming that a logarithmic distribution of head is applicable all the way to the well face (in other words, no turbulence is present) to (2) the drawdown in the well. These curves show a rapid drop in efficiency when discharges are increased beyond the critical discharge, and also show the importance of the well radius at higher rates of discharge.

Variations of Drawdown with Time

The foregoing discussion is based on a constant time; that is, all relationships are for a well pumped at a constant rate for a given time (1 hour in the example). In the example recharge from the river was occurring, and steady-flow conditions existed.

For the c[illegible] of a well in an infinite aquifer with neither barriers nor recharge, the [illegible]ression for drawdown in the well $s_w = BQ + CQ^n$ may be altered[9] to include the time factor by equating the formational term BQ to the drawdown as expressed by the simplified Theis equation

$$s_w = \frac{Q}{4\pi T}\left(2.3 \log \frac{4Tt}{r_w^2 S} - .577\right) + CQ^n \qquad (22)$$

in which S is the storage coefficient and r_w the "effective radius" of the well.

If we use Jacob's definition for effective radius, "the distance, measured radially from the well, at which the theoretical drawdown based on the logarithmic head distribution equals the actual drawdown just outside the screen," the value of r_w is unknown.

Reference to figure 2 and to the comparison of the empirical and theoretical methods of analysis shows that the distance from the well center to the point where the BQ and CQ^n portions of the drawdown are separated (r_B) occurs at the critical-flow boundary for a discharge Q_B. In the example the term r_B equals 0.69 foot. This seems a logical radius to use in the time equation in place of r_w, since losses beyond this point are expressed empirically as formational losses, and losses between r_B and r_n as "well losses" which are considered constant with time.

Although Jacob did not mention partial penetration, it is obvious that the equation must be altered if the pumped well penetrates only a part of the aquifer.

If the empirical expression is accepted, then the "well loss" CQ^n hypothetically occurs in a very narrow zone (between $r_B = 0.69$ foot and $r_n = 0.45$ foot). If r_B be considered the "effective radius" of the well the penetration factor would be applied only to the BQ term, using r_B in place of r_w in the Kozeny equation. If it be assumed that penetration effects are present between r_B and r_n, then the correction applies to both BQ and CQ^n, and r_n should be used in computing the correction. It should be noted that the Kozeny equation was derived for laminar flow and for a condition of steady flow. If the Kozeny equation is used for time problems, it is found that the BQ term is made up of two parts, a constant representing the penetration correction and a variable representing the formational loss which increases with time.

The theoretical drawdown distribution is considerably different than the $BQ + CQ^n$ empirical distribution. Until further analysis is made of the penetration problem, any attempts to relate step-test results to time problems for partially penetrating wells should be carried out with caution.

The empirical expression for drawdown in the well ($BQ + CQ^n$) is applicable to all types of problems for fully penetrating wells and for constant-time problems for partially penetrating wells, and it can be used within certain limits for problems involving partial penetration and time effects.

SUMMARY AND CONCLUSIONS

The important points covered in this paper may be summarized as follows:

1. On the basis of field data for several tests, it has been demonstrated that the empirical equation

$$s_w = BQ + CQ^n$$

9. Jacob, C. E., Drawdown test to determine effective radius of artesian well: Am. Soc. Civil Eng. Trans., vol. 112, paper no. 2321, p. 1055, 1947.

defines the total drawdown of a pumped well more closely than the [illegible]tion

$$s_w = BQ + CQ^2$$

proposed by Jacob.

2. A graphical method is presented for solving step tests according to the empirical equation $s_w = BQ + CQ^n$ by a simple plotting on log-log paper.

3. Computation of Reynolds numbers shows that, outside the well, laminar flow may occur at low pumping rates and turbulent flow at higher rates.

4. Analysis shows that the boundary between laminar and turbulent flow moves outward from the well face as discharge rates are increased.

5. An approximate equation has been written that evaluates the drawdown in the well in terms of laminar flow in the aquifer outside the critical radius. Turbulent flow from the critical radius to and through the well face, and frictional loss due to upward flow in the well.

6. Comparison of the log-log graphical method and the theoretical method is made as follows:

The $BQ + CQ^n$ method is very simple in application and allows for experimental errors. The exponent n is empirical and should not be confused with the theoretical (square) relationship of turbulent flow. The method is not applicable to problems of head distribution outside the well or for designing radii of wells. The term BQ carries the logarithmic distribution of head into the turbulent zone, so that the separation of the terms for aquifer loss and "well loss" is empirical and does not agree with theory. The approximate equation more nearly describes the true head distribution outside the well and should be used for this type of problem.

Comparison of the total drawdown obtained by the two methods shows close agreement at low and medium discharge rates but some deviation at higher rates. The deviation may be the result of erroneous assumptions or, an erroneous assumption in the empirical expression or may result from factors such as well development, not included in the theoretical analysis. In view of the fact that the deviation in total drawdown computed by the two methods is small at low and medium discharge rates and that uncertainty exists as to which method is in error, it is suggested that the empirical method be used because of its simplicity.

7. The bearing of radius on well design is more important than is indicated in current literature. Under certain conditions large savings in head loss can be made by using larger-diameter wells.

8. Efficiency of a well falls off rapidly as discharge is increased.

9. For fully penetrating wells step-test data may be used in problems having a variable time factor. For partially penetrating wells, the step-test results are satisfactory for constant-time problems, and can be used for variable-time problems under certain conditions.

10. The entire problem of effects of partial penetration should be investigated.

11. Other factors, such as the effects of development of a well, should also be investigated.

APPENDIX

Notation

B Head loss of formation per unit of discharge

C "Well-loss" constant for turbulent flow for a given well

C	“ -loss” constant for laminar flow for a given well
D	Aquifer constant for turbulent flow for a given well
E	Aquifer constant for turbulent flow for any well
F	Constant for head loss due to turbulent flow up the well
M	Head loss between the external boundary, r_e, and the inside of the screen, r_n, according to logarithmic distribution for laminar flow
N	Head loss between the turbulent-laminar boundary (r_t) and the inside of the screen (r_n) according to logarithmic distribution for laminar flow
Q	Discharge of well
Q_B	Discharge at which the actual turbulent-laminar flow boundary coincides with the empirical boundary r_B
Q_c	Critical discharge below which laminar flow prevails
S	Coefficient of storage
T	Transmissibility of aquifer; a property of the aquifer expressed as the quantity of water flowing through a vertical section of the aquifer of unit width, under a gradient of unity
a	Area
m	Thickness of aquifer
n	Unknown constant power relating discharge to “well loss”
r_1, r_2	Distance from well center to any point 1, 2, --
r_e	Effective distance from well center to boundary of zero drawdown
r_B	Radius at which turbulent and laminar losses are separated by the empirical equation $s_w = BQ + CQ^n$
r_n	Inside radius of well or well screen
r_t	Distance from well center to boundary between laminar and turbulent flow
r_w	“Effective radius” of well
dr	Increment of distance from well center
s	Drawdown at any point caused by pumping a well
s_w	Drawdown in the pumped well; at the end of any multiple step, s_w equals the sum of the incremental drawdowns for all steps
s_t	Head loss due to turbulent flow between the turbulent-laminar boundary (r_t) and the inside of the screen (r_n)
s_f	Head loss inside the well resulting from upward flow in the casing
s_1, s_2	Drawdowns at any point a distance r_1, r_2, -- from well center
$\Delta s_1, \Delta s_2$	Increments of drawdown resulting from changes in pumping rate
ds	Increment of drawdown
log	All logarithms are to the base 10
v	Velocity
θ	Porosity

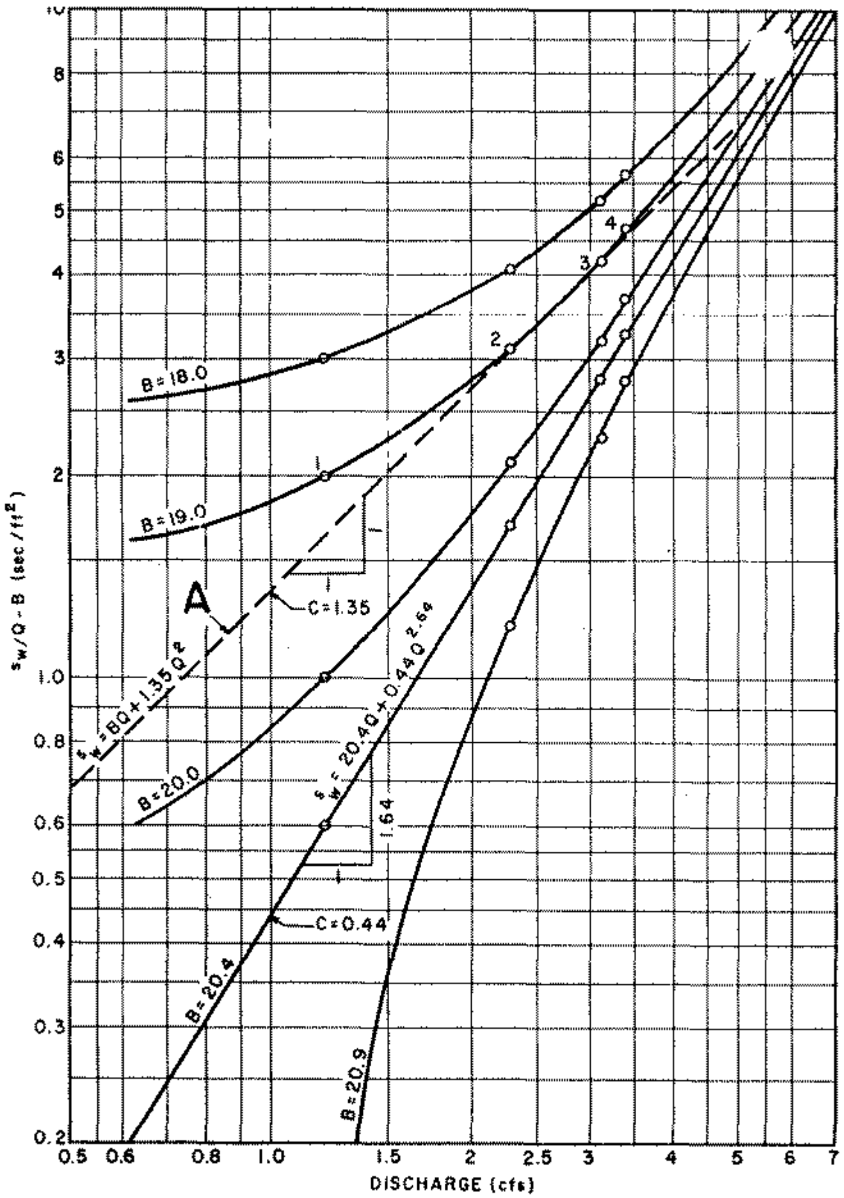

Figure 1.--Graphical solution of step test ($s_w = BQ + CQ^n$) and comparison with Jacob's solution ($s_w = BQ + CQ^2$). Data from Long Island test used by Jacob.

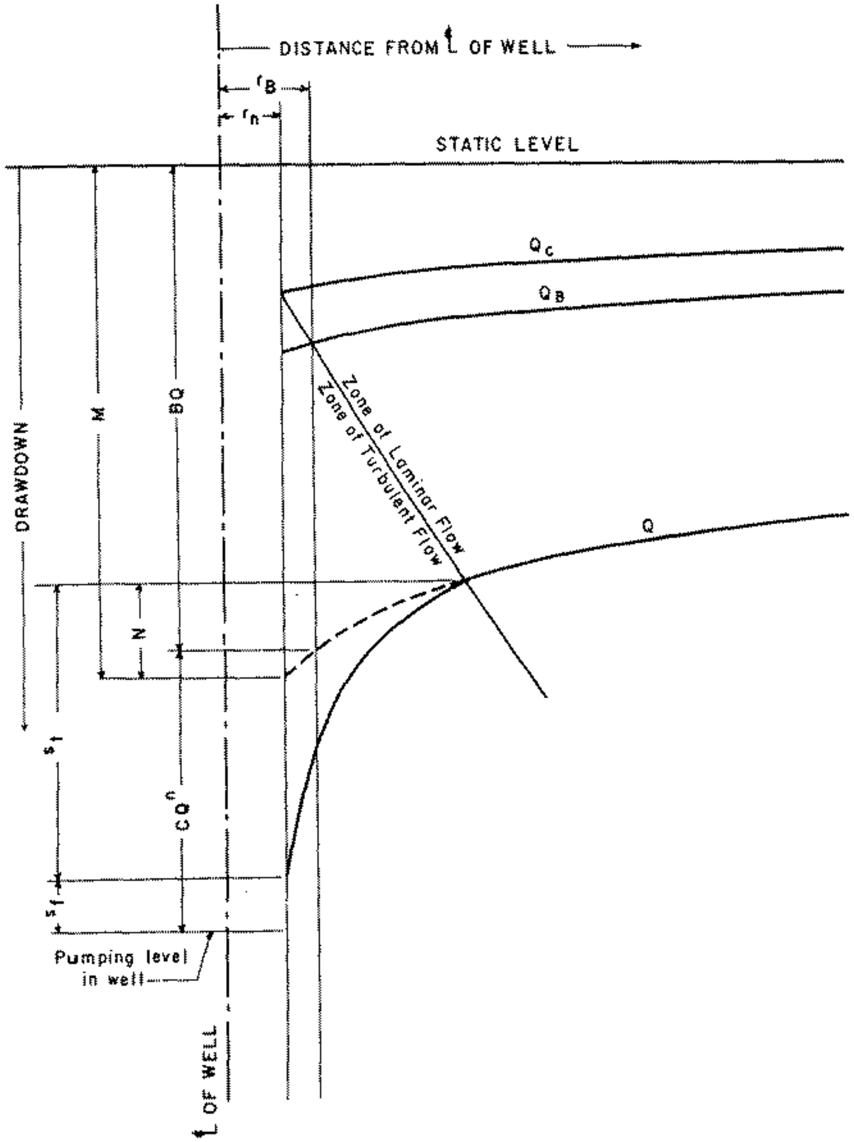

Figure 2. --Generalized section showing head distribution in and near a pumped well

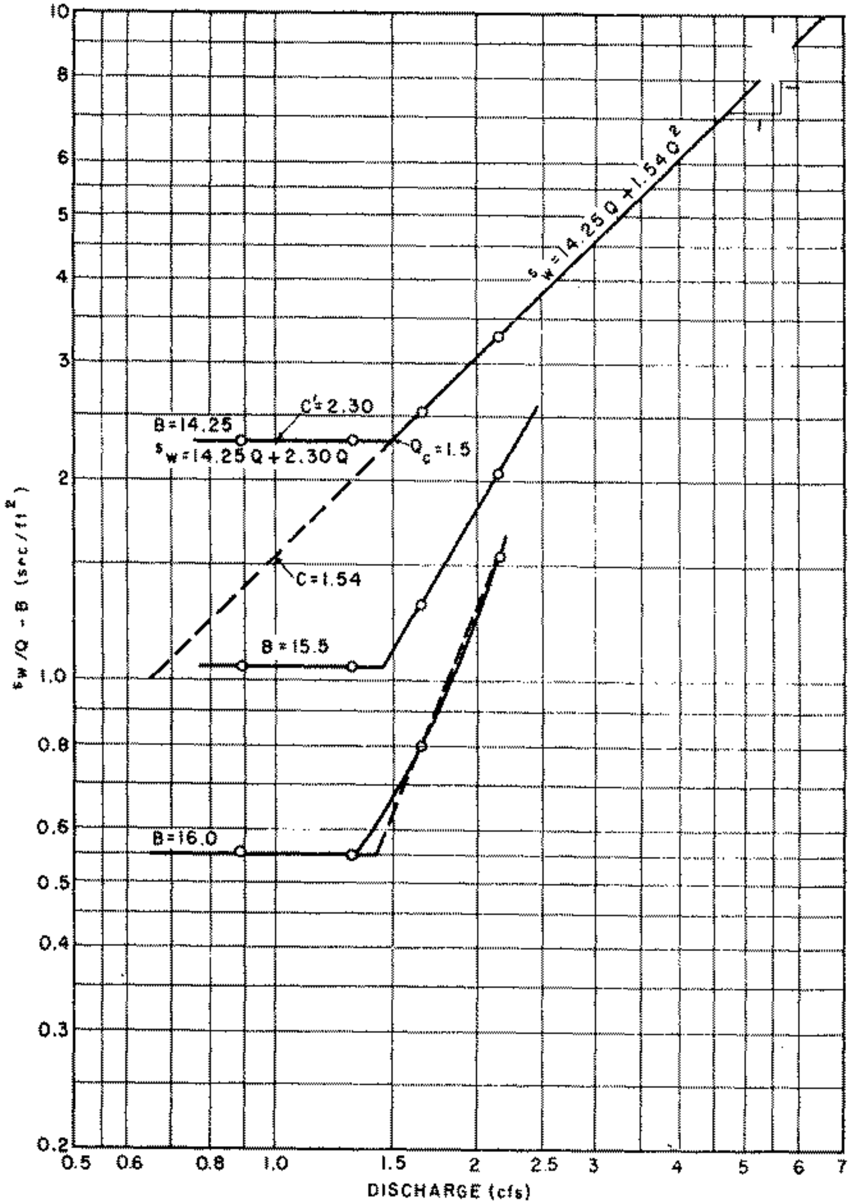

Figure 3. --Graphical solution of step test run in 1945, southwest of Louisville, showing discharges above and below critical discharge

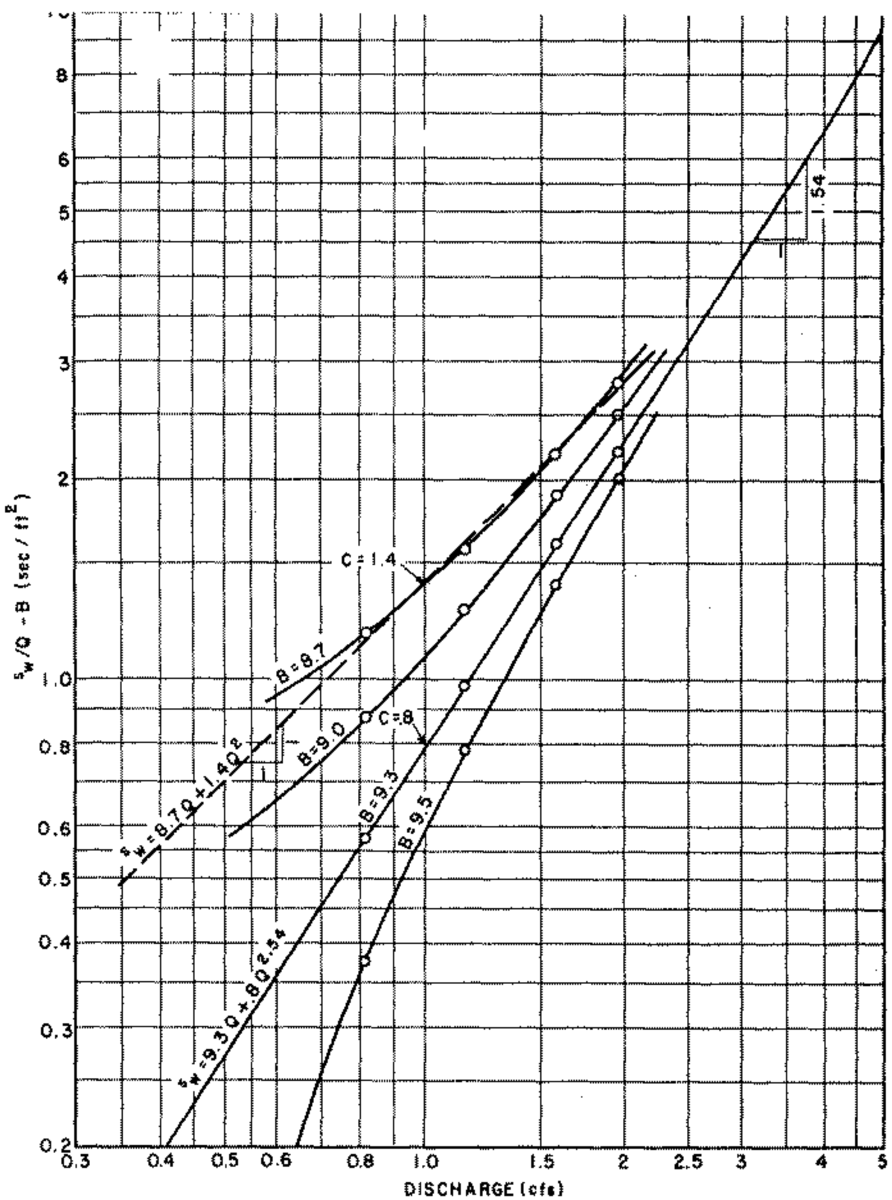

Figure 4.--Graphical solution of step test run in 1946, northeast of Louisville

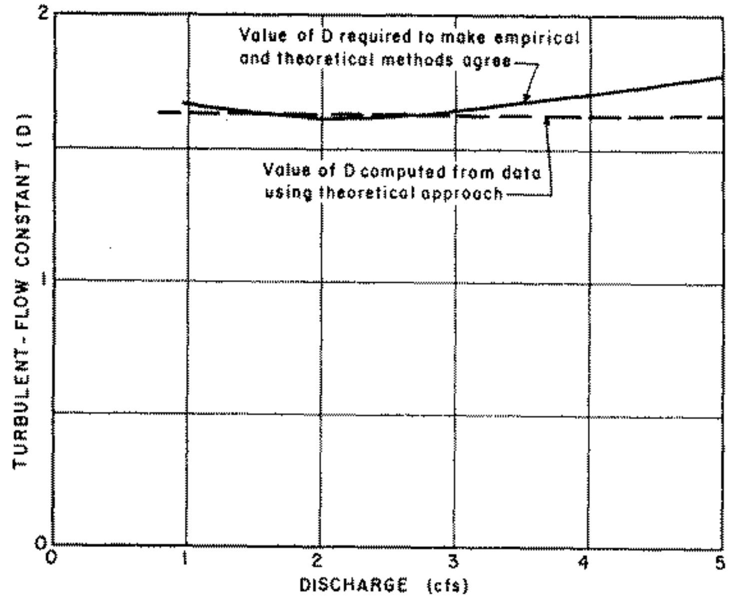

Figure 5.--Graph showing variation in turbulent-flow constant, D, required to make theoretical and empirical methods agree

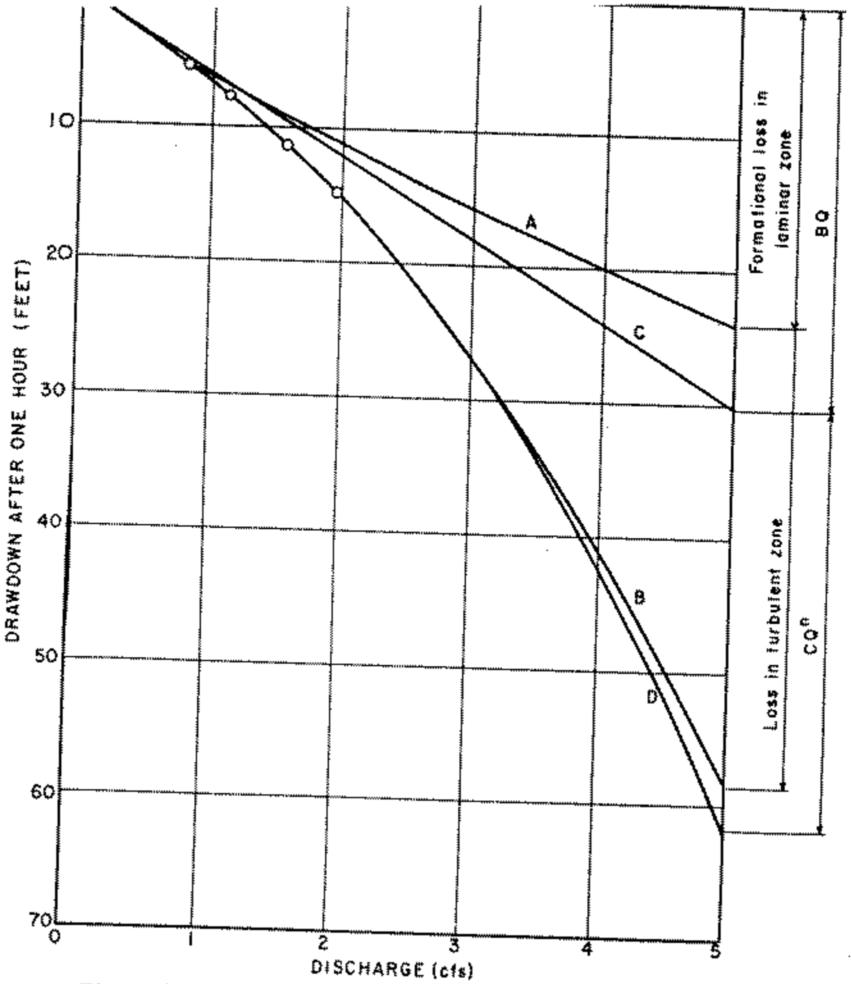

Figure 6.--Graph showing 1-hour discharge-drawdown relationship as computed by theoretical (B) and empirical (D) methods (northeast Louisville test, 1946; adjusted to full penetration); also shows proportion of head loss assigned to laminar and turbulent flow

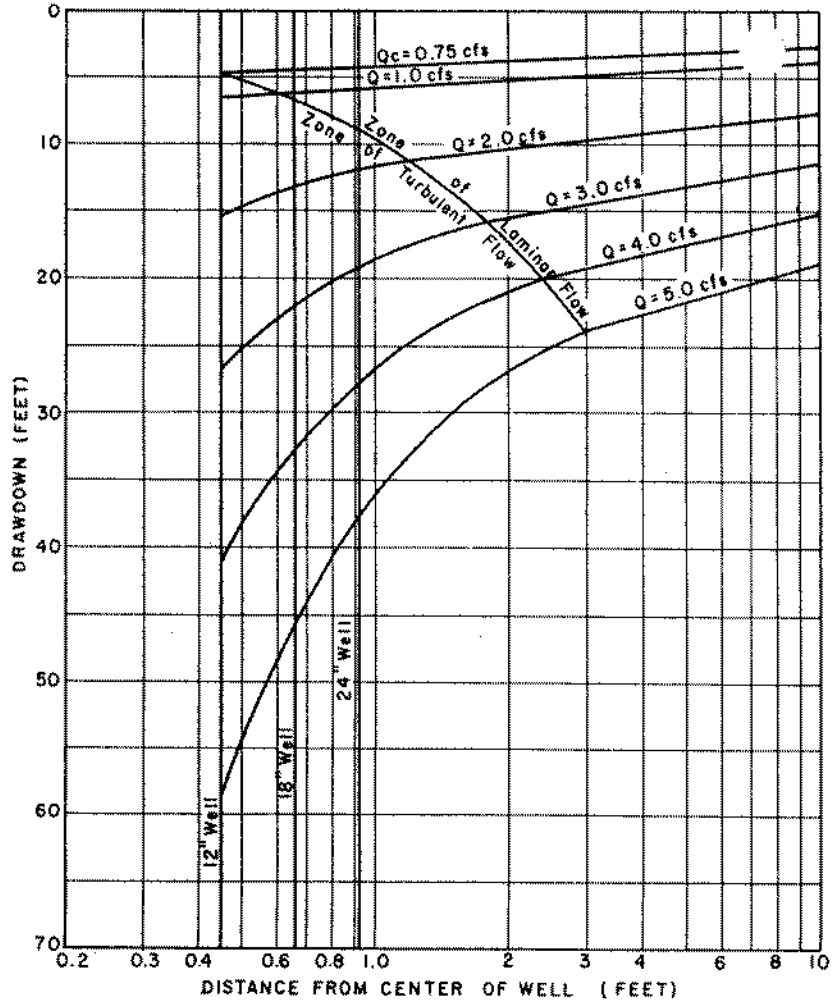

Figure 7.--Graph showing 1-hour head distribution near a pumped well for various pumping rates, (data from northeast Louisville test, 1946, adjusted to full penetration)

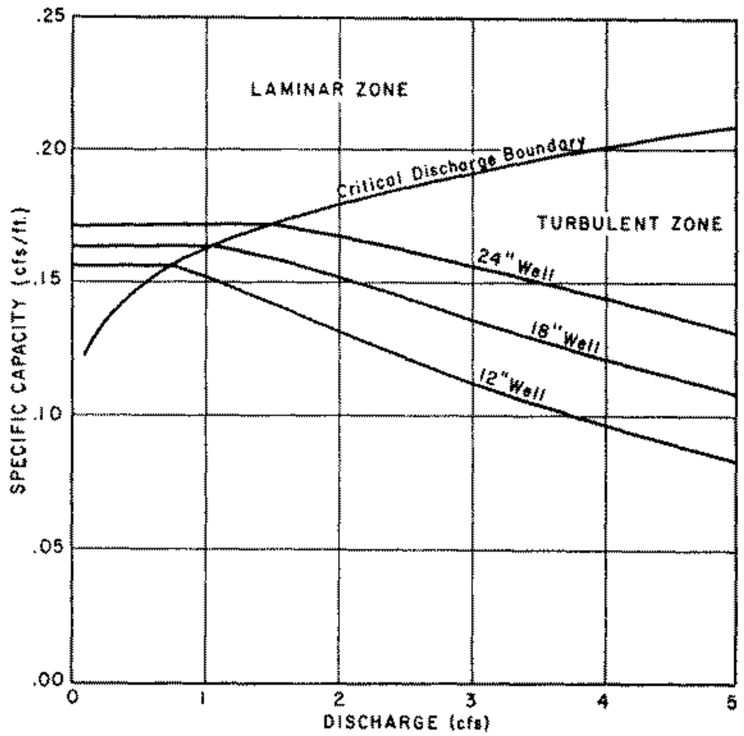

Figure 8.--Graph showing variation of 1-hour specific capacity with discharge and with well radius (northeast Louisville test, 1946, adjusted to full penetration)

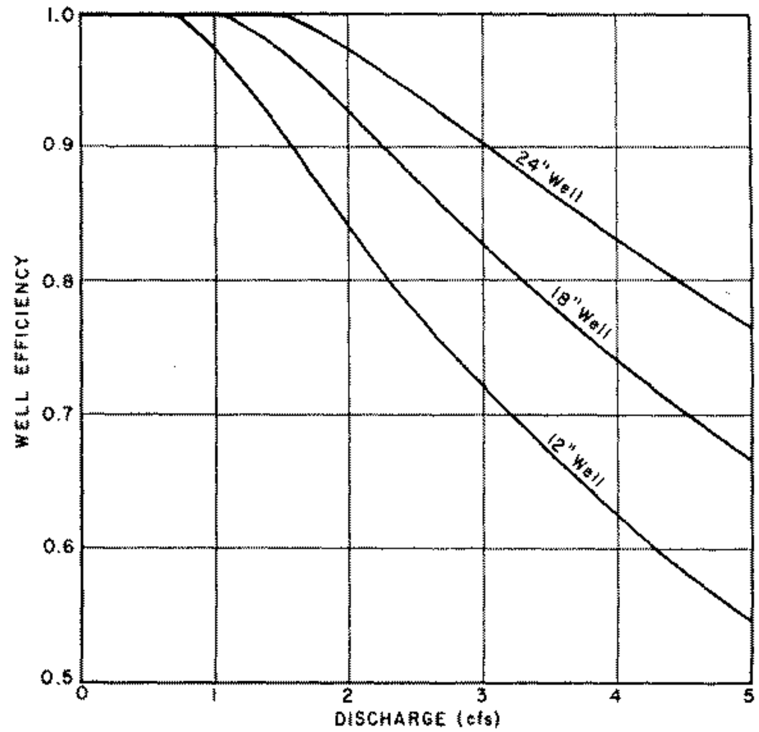

Figure 9.--Graph showing 1-hour well efficiency for wells of various diameters and various discharges (northeast Louisville test)